Phase diagram of Yang Mills theory in d=0,1,2 and application to black hole physics

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with M. Mahato and T. Morita (arXiv:0910.4526)
with T. Morita and S.R. Wadia (arXiv:1101.xxxx)
with H. Osono, (in progress)and
with P. Basu, T. Morita and S.R. Wadia (in progress)
Takeshi Morita (arXiv:1005.2181)

Motivation

Gauge theory

- Studying phase diagrams of large N gauge theories is important:
(a) Confinement/deconfinement transition
(b) Chiral symmetry breaking transition

- There are also specifically "large N" transitions
(c) GWW transition

- Methods of study:
weak 'tHooft coupling: Large N perturbation theory, nonperturbative (large D)
strong coupling: Lattice, Gauge-gravity.

Motivation

Gravity

- In gravity, there are interesting phase transitions in their own right:
(a') Hot gravitons (in AdS) $\rightarrow$ AdS Black holes (Hawking-Page)
(b') Black strings $\rightarrow$ Black holes (Gregory-Laflamme)
(c') AdS soliton $\rightarrow$ AdS Black hole (Witten,...)

- Conjectured correspondences (good evidence in special cases)
(a) Confinement/Deconfinement <-> (a') Hawking-Page
(c) GWW <-> (b') GL
etc.

Motivation

Gauge-gravity

- Correspondences are best studied for N=4 SYM in 4 dimensions (time x R^3 or time x S^3) <-> AdS_5 (Poincare) or AdS_5 (global), which are derived from a scaling limit of D3 branes.

- However, it is possible to extend these to lower Dp (p<3) branes, which yields the correspondence
d=(p+1) dim gauge theories (with D=9-p adjoint scalars) <-> AdS_{p+2} gravity
[cf. IMSY, Aharony-Marsano-Minwalla-Wiseman, Martinec-Sahakian,..., our works]

New technique

- On the gauge theory side, perturbation theory sometimes does not take us far, and nonperturbative techniques are required.
- We will discuss a new technique* to compute free energies and various order parameters in d dimensional (d=0,1,2) gauge theories. Consider a bosonic YM theory with action

$$S = \frac{1}{4}\int d^d x Tr\left(F_{\mu\nu}^2 + \frac{1}{2}\sum_{I=1}^{D} D_\mu Y^I D^\mu Y^I - g^2 \sum_{I,J} \frac{1}{4}[Y^I, Y^J]^2\right)$$

Can we treat the Y^4 term in a fashion similar to 4-fermi terms as in Gross-Neveu or NJL models?

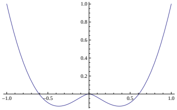
*Hotta-Nishimura-Tsuchiya, Mahato-Mandal-Morita

Large N_f

Recall Gross-Neveu:

$$S = \int d^2x \left(\bar{\psi}_i \partial_\mu \gamma^\mu \psi_i - g(\bar{\psi}_i \psi_i)^2 \right)$$

The technique to solve Gross-Neveu model is to introduce an auxiliary dynamical field ϕ, $g(\bar{\psi}_i\psi_i)^2 = \phi\bar{\psi}_i\psi_i - \phi^2/(4g)$ and integrate out the fermions to get



$$S_{eff}[\phi] = N_f \, \mathrm{logDet}(\gamma^\mu \partial_\mu + 2\phi) + \phi^2/(4g)$$

In the large N_f limit, $N_f g = \lambda$ fixed, the 1-loop term competes with the tree level term. Hence, a non-trivial value of the flavour-singlet condensate

$$< \phi > = \frac{2\lambda}{N_f} < \bar{\psi}_i \psi_i > \neq 0$$

appears at the new saddle point. [BCS, χSB, ...]

YM= Bosonic Gross-Neveu

- Can we write $Y^4 = -B^2/4 + BY^2$ etc. to get a non-trivial vacuum with $< Y^2 > \neq 0$?
What could Y^2 be? It can't be of the form $Tr[Y, Y]$ which trivially vanishes. It can be $Tr(Y^I Y^J)$, but we can't write $Tr([Y_I, Y_J]^2) = B_{IJ} Tr[Y^I Y^J] - B_{IJ}^2/4$ (single trace $\neq$ double trace).

- Turns out that by considering gauge-non-invariant, but SO(D)-invariant auxiliary fields, we CAN write

$$Tr[Y_I, Y_J]^2 \equiv -Y_a^I Y_b^J M_{ab,cd} Y_c^J Y_d^J = B_{ab} M_{ab;cd}^{-1} B_{cd} - 2i B_{ab} Y_a^I Y_b^I$$

where we have written $Y^I = Y_a^I \lambda_a$, and

$$M_{ab,cd} = -\frac{1}{4}\left\{ Tr[\lambda_a, \lambda_c][\lambda_b, \lambda_d] + (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d) \right\}$$

Now Y is only quadratic; integrating over Y, we get

Large D saddle point

$$Z = \int DA_\mu DB_{ab} \exp[-S_{eff}[A,B], \; S_{eff}[A,B] =$$

$$= \int d^d x [\frac{1}{4g^2} \left(F_{\mu\nu}^2 + B_{ab} M^{-1}_{ab;cd} B_{cd} \right)] + (D/2)\mathrm{logDet}(-D_\mu^2 \delta_{ab} + iB_{ab})$$

The idea now is to take a 'tHooft-like limit $D \to \infty, g^2 \to 0$ with $g^2 D = (\hat{g})^2$ held fixed. The determinant term will now compete with the tree level term, leading to a new large D saddle point for $< B_{ab} >= iM_{ab,cd} < Y_c^I Y_c^I >$ Note complex contour.

- In the examples we consider below, we will obtain saddle point values of the form $< B_{ab} >= i\Delta^2 \delta_{ab}$, which will imply dynamical generation of a condensate of the form

$$(1/D) < Y_a^I Y_b^I >= \Delta^2 \delta_{ab}$$

or, equivalently a mass gap $M_Y = \Delta$ (cf. the BY^2 term). In the large D saddle point, the field B_{ab} can be treated as classical, leading to a large D evaluation of $S_{eff}[A]$.

Other questions

- Volume dependence of results (e.g. nature of phase transitions) in gauge theories (see later).

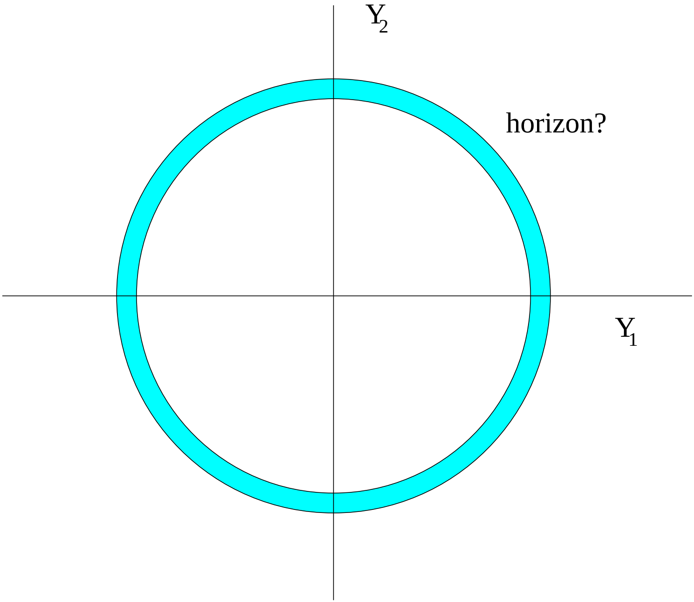
- The new technique can have potential applications to any model with $[Y^I, Y^J]^2$ interaction. E.g, the model

$$S = \int dt\, Tr \left((\partial_t Y^I)^2 - g^2 \sum_{I,J} \frac{1}{4} [Y^I, Y^J]^2 \right)$$

can be used to study issues related to the arrow of time [cf. Liu; Iizhuka & Polchinski, ...]

- Appearance of size (horizon?) $\langle Tr Y^I Y^I \rangle / (ND) \sim \Delta_0^2$. In fact, $\Psi(Y^2) \sim \delta(Y^2 - Y_0^2)$. (see figure)

- Saddle point configuration corresponds to black objects, with entropy $O(N^2)$. Origin in the Y^I quantum mechanics?


Y_2
horizon?
Y_1

Plan

- Introduction and Technique
- d=0
- d=1 and D1
- d=2 and D2
- Dynamical phase transitions
- Conclusions and open problems

d=0

No gauge fields!

$$
\begin{aligned}
Z &= \int dY^I \exp[-\frac{1}{4g^2} Tr \sum_{I,J} [Y^I, Y^J]^2] \\
&= \int DY^I_a DB_{ab} \exp[\frac{1}{4g^2} B_{ab} M^{-1}_{ab,cd} B_{cd} - \frac{i}{2g^2} B_{ab} Y^I_a Y^I_b] \\
&= \int DB_{ab} e^{-\mathcal{S}}, \mathcal{S} = \frac{1}{4g^2} B_{ab} M^{-1}_{ab,cd} B_{cd} + D/2 \mathrm{logdet}[B_{ab}] \qquad (1)
\end{aligned}
$$

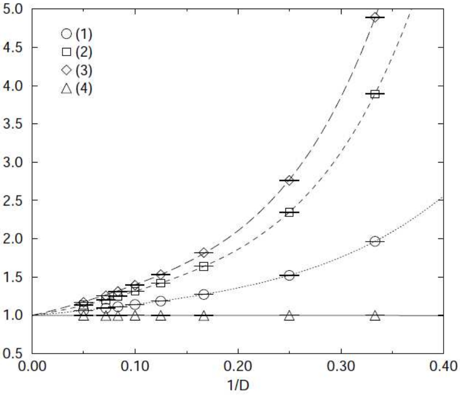
This can be computed at finite N, in a large D expansion! The leading term comes from the trace part $B_{ab} = B_0 \delta_{ab}$:

$$
\mathcal{S} = \frac{NB_0^2}{8\hat{g}^2} + \frac{(N^2-1)}{4} \log\left(-\frac{B_0^2}{\hat{g}^2 N}\right)
$$

where $(\hat{g})^2 = g^2 D$. At large N,

$$
F = -\frac{\log Z}{DN^2} = -\frac{1}{4} + \frac{\log 2}{4} + \frac{1}{D}\left(-\frac{5}{8} + \frac{1}{2}\log\frac{3}{2}\right) + O\left(\frac{1}{D^2}\right).
$$

d=0: comparison with numerics



The circles represent numerical values of 1/(DN) $< trY^I Y^I > /(\hat{g}/\sqrt{2})$ (extrapolated to $N = \infty$), while the dotted line represents the 1/D result discussed above. [The analytic result was also independently obtained by Hotta-Nishimura-Tsuchiya].

d=0

- Can compute F at finite N in a large D expansion.
- Solves bosonic IKKT.
- Full IKKT model can be solved for specific D's. [with Hiroshi Osono (in progress)]

d=1

This is the first non-trivial dimension involving a gauge field. Consider the size of the Euclidean dimension to be finite, β.

$$Z = \int DA_0 DY^I e^{-S},$$
$$S = \int_0^\beta dt\, Tr \left(\sum_{I=1}^{D} \frac{1}{2} \left(D_0 Y^I\right)^2 - \sum_{I,J} \frac{g^2}{4} [Y^I, Y^J][Y^I, Y^J] \right) . \quad (2)$$

Step 1: Wilson loop:
For finite β, can't gauge away A_0; fix gauge $\partial_t A_0 = 0$ [Aharony et al]

$$\Delta_{FP} = \exp[-S_{FP}], S_{FP} = N^2 \sum_{n=1}^{\infty} |u_n|^2/n$$

where $u_n = (1/N) Tr U^n, U = P \exp[i \oint dt A_0]$. Thus, A_0 reduces only to the Wilson loop (Polyakov loop).

$u_1 = 0$: centre symmetry unbroken ("confined" phase); $u_1 \neq 0$: centre symmetry broken ("deconfined" phase).

d=1: Integrate Y's

Step 2: Integrate out Y^I:
We show results only for the dominant mode $B_{ab}(t) = i\Delta^2\delta_{ab}$

$$\frac{D}{2}\log\left(\det\left(-D_0^2+\triangle^2\right)\right) = \frac{DN^2\beta\triangle}{2} - D\sum_{n=1}^{\infty}\frac{x^n}{n}|u_n|^2.$$

Combining with the classical B^2 term, and Δ_{FP} we get

$$\frac{\mathcal{S}(\triangle,\{u_n\})}{DN^2} = -\frac{\beta\triangle^4}{8\tilde{\lambda}} + \frac{\beta\triangle}{2} + \sum_{n=1}^{\infty}\left(\frac{1/D - x^n}{n}\right)|u_n|^2.$$

where $\tilde{\lambda} = \lambda D = g^2 ND$ is the large D 'tHooft coupling.

d=1: Large D saddle point

Step 3: Evaluate Δ at the saddle point

$$\triangle_0(\{u_n\}) = \tilde{\lambda}^{1/3}\left(1+\frac{2}{3}\sum_{n=1}^{\infty}\bar{x}^n|u_n|^2\right)+\cdots,$$

where $\bar{x} = \exp[-\beta\tilde{\lambda}^{1/3}]$.
Step 4: Put this back in $\mathcal{S}[\Delta,\{u_n\}]$:

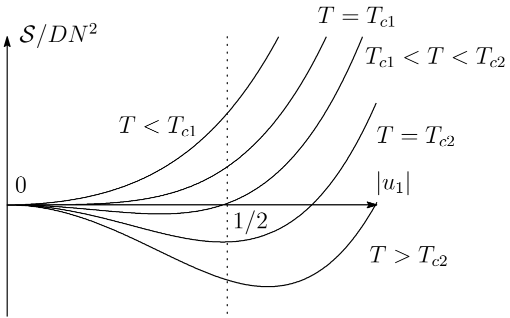
$$\frac{\mathcal{S}(\{u_n\})}{DN^2} = \frac{3}{8}\beta\tilde{\lambda}^{1/3} + {\color{red}a_1|u_1|^2 + b_1|u_1|^4} + \sum_{n=2}^{\infty}a_n|u_n|^2+\cdots,$$

$$a_n = \frac{1}{n}\left(1/D-\bar{x}^n\right),$$

$$b_1 = \frac{1}{3}\beta\tilde{\lambda}^{1/3}\bar{x}^2, \qquad (3)$$

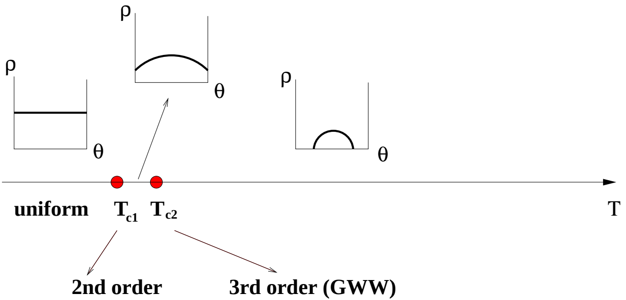
where the $\cdots$ involve other u_n^4 terms for $n>1$, which are down at large D.

d=1: Landau-Ginzburg



As T crosses T_{c1}, u_1 becomes tachyonic and there is a second order phase transition which signals an onset of non-uniformity in the eigenvalue distribution $\rho(\alpha)$. At $T = T_{c2}$, characterized by a potential minimum at $|u_1| = 1/2$, a gap develops in the eigenvalue distribution, signalling a GWW transition.

d=1: phase diagram



	T_{c1}	T_{c2}	R^2	F_0
Numerical result	0.8761	0.905	2.291	6.695
Leading large-D result	0.947	0.964	2.16	7.02
Large-D including $1/D$ effect	0.895	0.917	2.28	6.72

2nd and 3rd rows are our results, with $D = 9$ (10-dimensional YM theory compactified to d=1). Numerical results are from Nishimura and Kawahara. The agreement between the 3rd row and the 1st row are within 1% (which is $1/D^2$).

d=1: chemical potential [Takeshi]

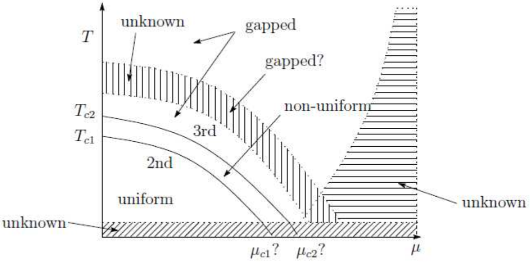


Figure 1: Phase diagram of the one dimensional gauge theory in $\mu - T$ space from the $1/D$ expansion. Three (uniform, non-uniform and gapped) phases exist and the orders of the phase transitions between them are second and third. In the shaded regions, it is difficult to analyze the model through the $1/D$ expansion. In the horizontal shaded region (very high chemical potential region $\mu/\tilde{\lambda}^{1/3} \gg (k_2 T/\tilde{\lambda}^{1/3})^{1/4}$), the expansion does not converge because of the existence of the light mass modes. In the inclined shaded region (very low temperature region $T/\tilde{\lambda}^{1/3} < D^{-\gamma}$), the expansion is not valid, since the effective coupling $\tilde{\lambda}/T^3$ becomes too strong. The analysis in the vertically shaded region is also difficult, since the Wilson loop operators are highly interacting (intermediate region). However we can guess that the vertically shaded region will be gapped phase. See Table 1 also.

Gravity correspondence: D1 branes

D1 branes= 2d SYM; wrap D1 on L. Also curl the Euclidean time direction into a circle of length β.
There are various possible BC's for the fermions along β, L. (AP, P) corresponds to a black string. In terms of dual D0 branes, we have a Gregory-Laflamme transition.

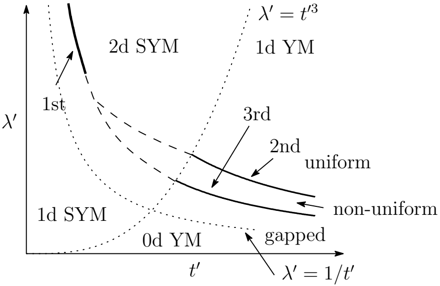


Figure: $\lambda' = \lambda_2 L^2, t' = L/\beta_2$. Below $\lambda' = t'^3$, the temporal KK modes (and fermions) can be ignored, ($\Rightarrow$ $d = 1$ YM). Below $\lambda' = 1/t'$, the spatial KK modes can be ignored ($\Rightarrow$ d=1 SYM). The overlap of these 2 regions $\Rightarrow$ d=0 YM. The two phase transition lines below $\lambda' = t'^3$ are given by $\lambda' t' = 1/T_{c1}^3$ and $\lambda' t' = 1/T_{c2}^3$. A similar phase structure was earlier inferred in [Kawahara] on the basis of numerical analysis.

d=2

Consider $d = 2$ Euclidean YM theory with D ajoint scalars, compactified on a 2-torus T^2.

$$S = \int_0^\beta dt \int_0^L dx \, Tr \left(\frac{1}{2g^2} F_{01}^2 + \sum_{I=1}^{D} \frac{1}{2} \left(D_\mu Y^I \right)^2 - \sum_{I,J} \frac{g^2}{4} [Y^I, Y^J][Y^I, Y^J] \right)$$

We now have two Wilson lines $U = P \exp[i \oint^\beta A]$ and $V = P \exp[i \oint^L A]$ along the two cycles. There are now possibly 4 or more phases, corresponding to whether TrU, TrV are zero or non-zero and whether a non-zero Wilson line can exist in 2 distinct phases (non-uniform vs gapped eigenvalue distribution).

d=2: small L

- For small enough L, the problem reduces to $d = 1$, with A_1 turning into an extra Y, which we have solved above.
- Note: large N volume independence arguments. In the centre symmetric phase (Tr V=0: uniform eigenvalues), KK reduction does not work in the usual fashion since new soft modes, with mass $\sim 1/(NL)$, appear. However, for small enough L, eigenvalues of A_1 are clumped near 0 (this is consistent with eigenvalues of A_0 getting more and more clumped at low enough β) hence centre symmetry along L is broken (Tr $V \neq 0$). Hence KK reduction works along L and the problem simplifies to the $d = 1$ model.

d=2: large L

Need to evaluate the 1-loop effective action

$$S^{(1)}(A_\mu, \Delta) = \frac{D}{2} \log\det\left(-D_\mu^2 + \triangle^2\right)$$

where $B_{ab}(x,t) = i\Delta^2\delta_{ab}$ is, as usual, the dominant mode at large D. Under the assumptions $L\Delta \gg 1, \Delta \gg \sqrt{\tilde\lambda}$, it turns out that the Wilson line V decouples from the dynamics, yielding (cf. SZ,AMMW,BEW)

$$S/DN^2 = \int_{-\infty}^{\infty} dx \left[\frac{1}{2N} Tr\left(|\partial_x U|^2\right) - \frac{\xi}{N^2}\left|TrU\right|^2\right].$$

where

$$\xi = \sqrt{\frac{\triangle_0}{2\pi\tilde\lambda^2\beta^3}}\, e^{-\triangle_0\beta}$$

and Δ_0 is an analog of Λ_{QCD}

$$\triangle_0 = \sqrt{\frac{\tilde\lambda}{2\pi}\log\left(\frac{2\pi\Lambda^2}{\tilde\lambda}\right)} + \cdots\,,\ \tilde\lambda = (2\pi\Delta_0^2)/\log(\Lambda^2/\Delta_0^2)$$

d=2: large L phase transition

The double trace action was analyzed in [Semenoff-Zarembo, Basu-Ezhuthachan-Wadia], using the eigenvalue density

$$\rho(\theta, x) = \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \theta_i(x))$$

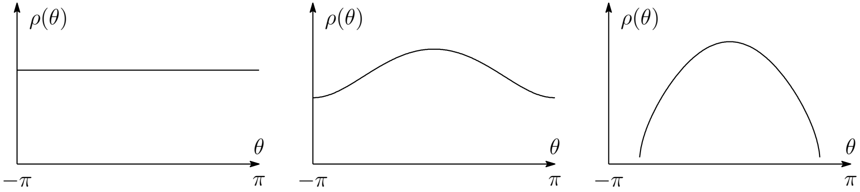
The hamiltonian becomes (at large N)

$$H = \int d\theta \left(\frac{1}{2} \rho v^2 + \frac{\pi^2}{6} \rho^3 - \xi \left| u_1 \right|^2 \right) .$$

where $v = \partial_\theta \Pi$. The hamiltonian admits x-independent solutions

$$\rho(\theta) = \frac{\sqrt{2}}{\pi} \left(\sqrt{E + 2\xi \rho_1 \cos\theta} \right)$$

The eigenvalue density can be uniform, non-uniform or gapped, for various ξ-values.



d=2: Landau-Ginzburg potential

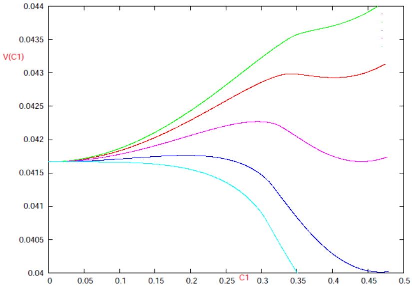
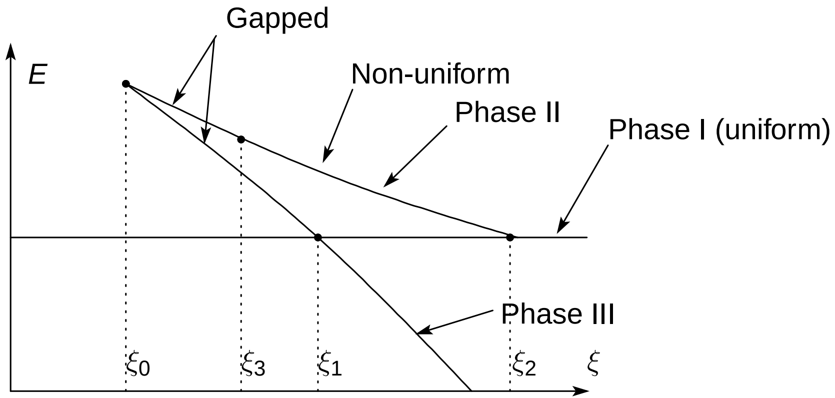


Figure 1: Plot of $V(C_1)$ with C_1 with $\xi = 0.22$, $\xi = 0.23$, $\xi = 0.237$, $\xi = 0.245$ and $\xi = 0.25$, with value of ξ increasing from the top curve to the bottom.

Here C_1 is roughly $< TrU >$ (in a static phase), and $V(C_1)$ can be regarded as an on-shell evaluation of the action S in the previous slide. There is a clear first order phase transition.

d=2: Stability and order of transition



Energy vs ξ for three types of eigenvalue distribution of the Wilson line U. ξ is a monotonically increasing function of T. Note the 1st order transition at ξ_1.

d=2: phase diagram

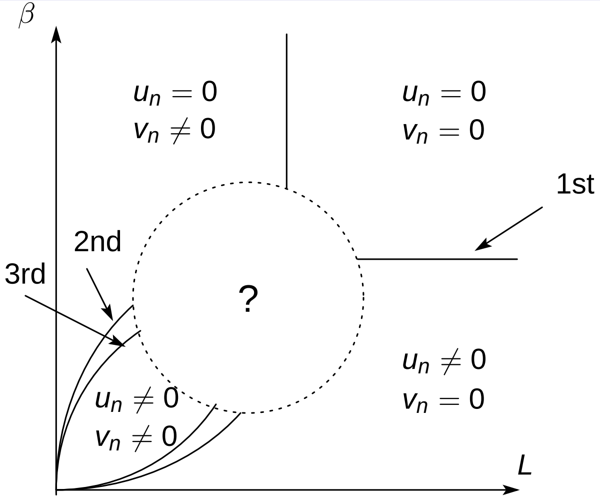
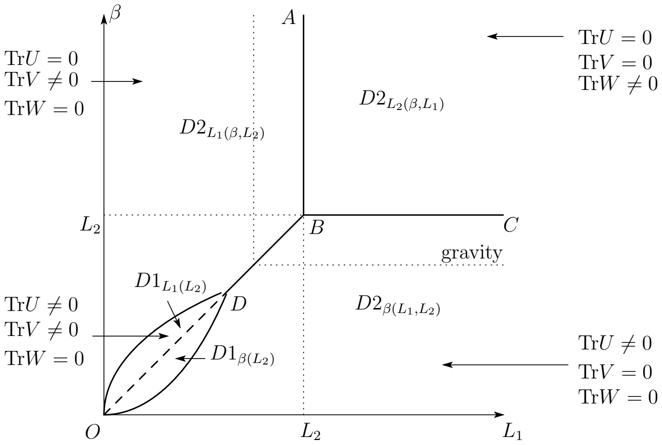


Figure: Phases at small and large L.

Gravity correspondence: D2 branes

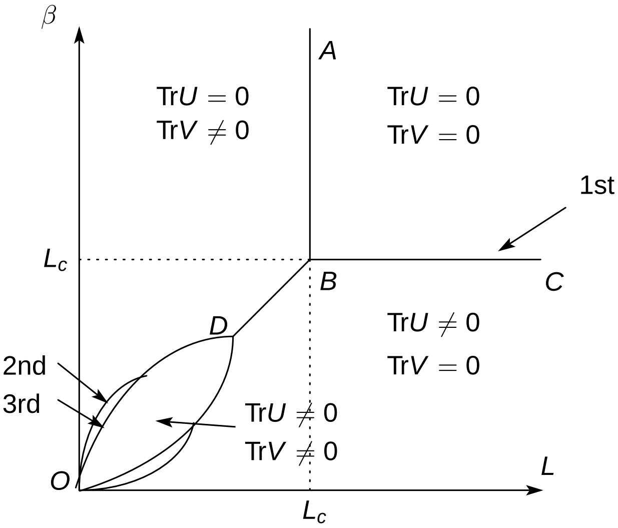
- To get a gravity dual of $d = 2$ bosonic YM, start with D2 branes= 3d SYM on T^3 with radii β, L_1, L_2.
- Consider AP b.c. for fermions along L_2. For small enough L_2 the corresponding KK modes and all fermions decouple $\Rightarrow$ $d = 2$ YM.
- However, for very small L_2, the gravity analysis is not reliable; hence L_2 cannot be taken too small, $\Rightarrow$ fermions persist.
- Phase diagram depends on fermion boundary conditions along β, L_1: (P,P), (AP, P), (P, AP), (AP,AP).
- Gravity solutions (phases) include D0, D1 and D2 branes (smeared/ localized) and AdS solitons which are double Wick rotations of these.

D2: (AP, AP, AP)



β
A
$\mathrm{Tr}U = 0$
$\mathrm{Tr}V = 0$
$\mathrm{Tr}W \neq 0$
$\mathrm{Tr}U = 0$
$\mathrm{Tr}V \neq 0$
$\mathrm{Tr}W = 0$
$D2_{L_2(\beta,L_1)}$
$D2_{L_1(\beta,L_2)}$
L_2
B
C
gravity
$D1_{L_1(L_2)}$
D
$D2_{\beta(L_1,L_2)}$
$\mathrm{Tr}U \neq 0$
$\mathrm{Tr}V \neq 0$
$\mathrm{Tr}W = 0$
$D1_{\beta(L_2)}$
$\mathrm{Tr}U \neq 0$
$\mathrm{Tr}V = 0$
$\mathrm{Tr}W = 0$
O
L_2
L_1

d=2: Combining gauge theory & gravity-extrapolated



d=2: Dynamical transitions

- We derived the large L effective action above. By flipping $t \leftrightarrow x_1$, we get the following effective action at large β (low temperature)

$$S(A)/DN^2 = \int_0^\infty dt \left(\frac{1}{2N} \mathrm{Tr} \left(|\partial_t V|^2 \right) + \sqrt{\frac{\triangle_0}{2\pi\tilde{\lambda}^2 L^3}} e^{-\triangle_0 L} \left| \frac{1}{N} TrV \right|^2 \right)$$

where $V(t) = P \exp[i \int A_1(x,t)dx]$.

- The static solutions, as mentioned before, are given by uniform, non-uniform and gapped eigenvalue distributions. The stability of these depends on the value of L.

- By using the above action, we can consider dynamical transitions between these phases, which would include gauge theory duals of dynamical Gregory-Laflamme transitions.

Gapless $\rightarrow$ gapped

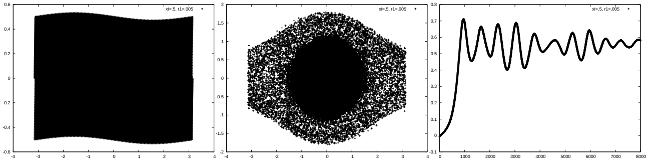


Figure: The figure on the left shows a slightly perturbed gapless distribution at $t = 0$. The figure in the middle shows a nearly gapped distribution (t=8000). The figure on the extreme right depicts $\rho_1(t)$ as it changes from 0 at $t = 0$ to 0.55 at $t = 8000$

Gapless $\rightarrow$ gapped: density plot

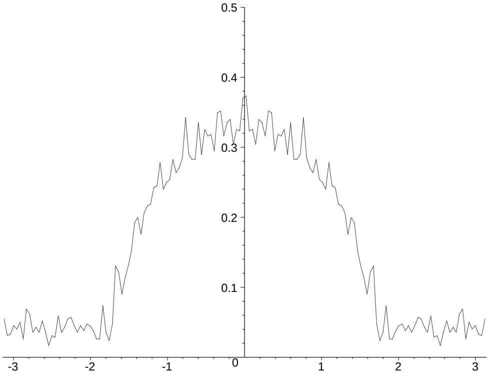


Figure: Coordinate space fermion distribution corresponding to the central figure of Fig 4. The ‘waist’ does not vanish at very large times. cf. Horowitz-Maeda conjecture: ‘no naked singularity’.

Gapped $\rightarrow$ gapless

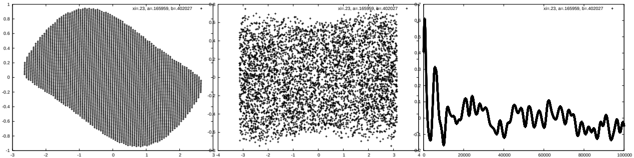


Figure: The figure on the left shows a slightly perturbed gapped distribution at $t = 0$. The value of ξ is 0.23. The figure in the middle shows a gapless distribution at $t = 10000$. The figure on the extreme right depicts $\rho_1(t)$ as it changes from 0.5 at $t = 0$ to 0 at $t = 8000$

Open problems and work in progress

- Fermions [work in progress with Hiroshi Osono]. Schematically,

$$\psi^2 Y + Y^4 = BY^2 + B^2 + \psi^2 Y$$
$$= B(Y + 1/(2B)\psi^2)^2 - \psi^4/(4B) = B(\tilde{Y})^2 - F^2/(4B) + \psi^2 F$$

$\Rightarrow$ SSB of SO(D).

- Higher dimensions ($d \geq 3$). In addition to $\log(D_\mu^2 + B)$, the kinetic term $F_{\mu\nu}^2$ plays an important role.
- Dynamical transitions: equilibration, time arrow
- $\langle TrY^I Y^I \rangle/(ND) \sim \Delta_0^2$. In fact, $\Psi(Y^2) \sim \delta(Y^2 - Y_0^2)$. Appearance of size (horizon?). Need to compute Wilson line in the bulk to compute the location of horizon.
- Saddle point configuration corresponds to black objects, with entropy $O(N^2)$. Origin in the Y^I quantum mechanics? Splitting of the $O(N^2)$ level....